Electrical Systems

Hasan A. Poonawala
Fall 2020

Contents

1 Introduction

Electrical circuits combine different elements that generate, store, and consume electrical energy in order to do useful work. Electrical energy can be transmitted by moving particles that have a negative or positive charge. Current electricity is electrical energy that flows from one place to another, usually through a wire. In most modern electrical systems, these particles are electrons, which are negatively charged particles.

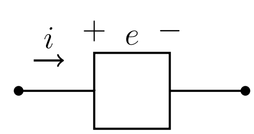
For example, your cellular phone contains a battery that operates as a source of electrons. Whenever we switch our phones on, these electrons flow through the phone as a current, along desired pathways, providing the energy that illuminates the screen, powers the flash, creates electromagnetic signals for communication, and so on. However, we don't use our flash all the time, our screen is not on all the time, and we aren't downloading videos all the time. In other words, the flow of current is not constant, but changes with time.

Managing this time-varying flow of current becomes important. We wouldn't buy a phone with a screen that dims every time we start recording a video because we didn't figure out how to ensure more activities automatically create a higher flow of current. Alternatively, if we don't manage the flow of current, and ask for more current than required, we waste battery charge and risk over-heating the phone. Therefore, we need predictive models that tell us how different choices (unlocking our screen, taking a picture with the flash on) will affect the flow of energy (current) through the rest of the system.

2 Circuit Elements

To get current to flow, thereby transmitting electrical energy to where it is needed, we must create a difference in electrical potential between the source of electrons and the destination. This difference in potential that is inducing a flow of current is known as a **voltage difference**.

We assume that current flows from one elements to another via a physical connection known as a terminal. Each circuit element then consists of two terminals, so that current flows into one terminal and out the other, and a voltage difference exists across these terminals.



Circuit element with voltage difference e and current i across terminals (black dots).

We have the variables:

Voltage e in Volts (V)

Current i in Amperes (A)

Charge q in Coulumbs (A $\cdot$ s)

These variable are related as follows. The current flowing through a point is the rate of change of charge at that point:

$$i = \frac{dq(t)}{dt},$$

so that

$$q(t) = \int_0^t i(s)ds.$$

The power P consumed by an element is $P = ei$ which has units of Watts (W = V $\cdot$ A).

The energy supplied to an element is $\int_0^t P(s)ds$, which has unit Joules (J = W $\cdot$ s).

So far, we have related current i and charge q, and defined power and energy in terms of i, e. These relationships don't depend on the circuit element.

What distinguishes different elements is the relationship between the voltage e across them and current i flowing through them.

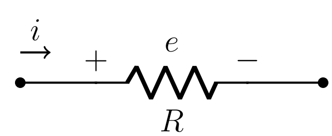
We look at four common elements:

1. Resistor
2. Capacitor
3. Inductor
4. Current/Voltage Sources

2.1 Resistor

Resistors possess an algebraic relationship between e and i. A linear resistor is one where this relationship is linear:

$$e = iR,$$



where the linear coefficient R with unit ohms (Ω) is called the resistance of the element.
The power consumed by a resistor is then $P(t) = e(t)i(t) = i^2(t)R = e^2(t)/R$.

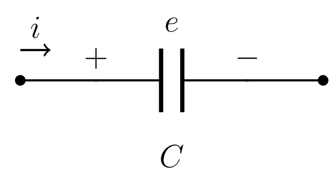
This power consumption is often referred to as i^2R losses, and shows up as heat. If you high current through a highly resistive element, you will see the element heat up quickly, melting nearby objects. This phenomenon has destroyed many household appliances.

2.2 Capacitor

Capacitors possess an algebraic relationship between voltage across its terminals and the *charge* stored in the capacitor. For a linear capacitor,

$$q(t) = Ce(t),$$

where C is the capacitance which as units farad (F). Often, the values of the capacitance are of the order of 10^{-6}, so the units micro-farads (μF) are used.



This algebraic relationship between e and q leads to a differential relationship between e and i:

$$e = \frac{1}{C}\int_0^t i(s)ds,$$

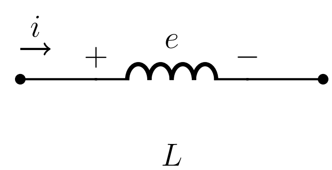
or

$$i(t) = C\dot{e}(t).$$

The energy stored in a capacitor is $Ce^2/2$.

2.3 Inductor

An electric current creates a magnetic field. Changing flux fields induce electrical potentials in conductors. Therefore, a time-varying current through a conductor will induce a potential field in itself, and this potential is proportional to the rate of change of current.



For a current flowing through an infinitely straight wire, the magnetic flux it produces doesn't end up passing through much of the wire. Instead, consider a coiled wire. The flux produced in one part of the wire passes through a significant portion of the rest of the wire, so that there's a strong link between the flux produced by a current in the wire and the flux passing through the wire.

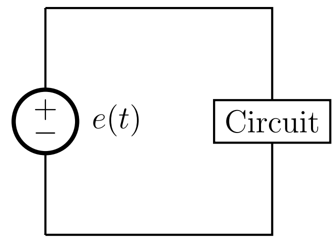
The net induced potential due to time-varying currents will depend on the geometry. We model the strength of the coupling between the flux produced by a conductor and the amount of that flux passing through it by the inductance L, with units henries (H). Some geometries create a linear relationship, so that

$$e = L\frac{di}{dt}.$$

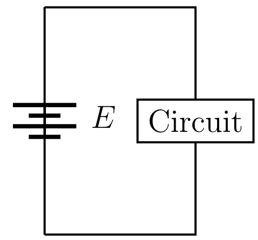
The energy stored by an inductor is $Li^2/2$.

2.4 Sources

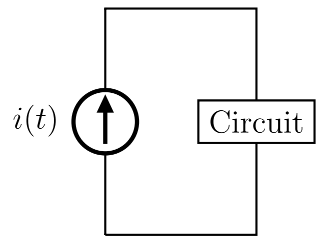
A voltage source maintains a certain voltage $e(t)$ across its terminals.



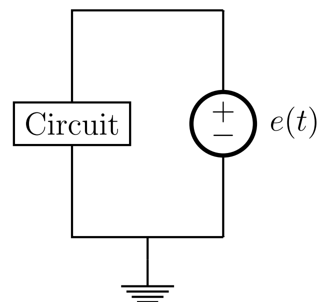
Batteries are examples of nearly-constant voltage sources. However, they cannot supply current indefinitely, as their charge is eventually depleted.



A current sources ensures that a certain current $i(t)$ flows through it.



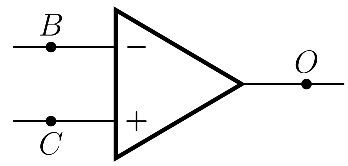
A ground represents a point of zero voltage.



3 Operational Amplifiers

An operational amplifier, or *op-amp* for short, is a circuit element with three terminals. There are many flavors of op-amps, we focus on ideal voltage amplifiers. The physics of this type of operational amplifier essentially creates an algebraic relationship between the voltages at the three terminals. Note that this behavior unlike other circuit elements that relate current and voltage across a pair of terminals.

An op-amp has three terminals, as shown to the right. Two are labeled $-$ and $+$ respectively, with the third unlabeled terminal – known as the output terminal – emerging from a corner of the triangular symbol. These terminals are connected by ideal conducting wires to points B, C and O belonging to some larger circuit (not shown).



The voltages at these three points satisfy the relationship

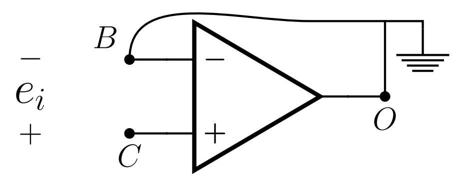
$$e_O = A(e_C - e_B),$$

where A is the gain of the amplifier, and is positive and typically larger than 10^5.

If we connect B and O, and apply a voltage e_i across B and C, then $(e_O - e_B) = Ae_i$. If $e_B = 0$, then

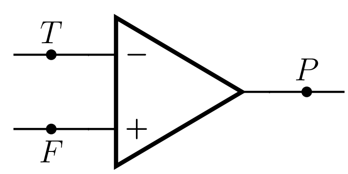
$$e_O = Ae_i,$$

indicating why the element is an ‘amplifier’,



For the operational amplifier to the right, with renamed points T, F, and P, on the wires leading out from the operational amplifier, we would obtain

$$e_P = A(e_F - e_T).$$



In summary, when an operational amplifier is placed in a circuit, we can write down an algebraic relationship between the voltages at its three terminals that depends on the gain of the op-amp. This idea is useful for circuit analysis, much like we can relate currents and voltages across the two terminals of other circuit elements.

3.1 Virtual Short

Rewrite $e_O = A(e_C - e_B)$ as

$$e_B = e_C - \frac{1}{A}e_O.$$

When A is very large, $1/A \approx 0$, so that $e_B \approx e_C$. This situation creates a *virtual short.*The virtual part comes from the fact that B and C have the same voltage *as if they were virtually connected by a wire*, but no wire exists so *there is no current flow* from B to C.

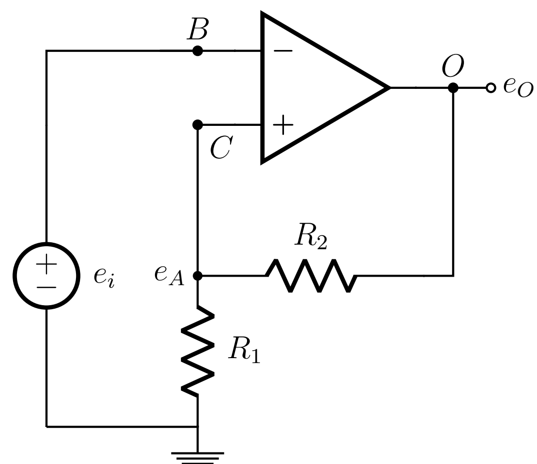
To the right, even though B and C are not connected, the voltages at these two points are the same!

Virtual short when $A \to \infty$
$e_B \approx e_C$, but e_O is unrelated to these two voltages.

Therefore, high-gain voltage-based operational amplifiers are a way to keep two different points in a circuit at the same voltage without allowing current to flow between them.

Consider the circuit to the right. We have $e_O = A(e_A - e_i)$, or

$$e_A = \frac{e_O}{A} + e_i.$$



KCL at A:,

$$-\frac{1}{R_1}e_A = \frac{e_A - e_O}{R_2},$$

so that

$$e_O = \frac{R_1 + R_2}{R_1}e_A$$

. This KCL is simply another way to derive the same result using the voltage divider rule:

$$e_A = \frac{R_1}{R_1 + R_2}e_o.$$

As $A \to \infty$, by the virtual short concept, $e_i = e_A$. Therefore,

$$e_O = \frac{R_1 + R_2}{R_1}e_i \text{ when } A \to \infty.$$

4 Circuit Analysis

Our goal is to examine the dynamic behavior of electrical systems. We typically apply an input, in the form of a voltage/current source, and we are concerned with some output variable of interest. For example, when we turn a constant voltage source on, will the current that flows through a motor (a resistance) in the long run be enough to drive an electric car? Will there current through another resistance destroy that element?

So far, we've considered the relationship between a voltage and current tied to a single circuit element. If we apply a constant voltage source across these elements, we would know the current flowing as a function of time by using the voltage-current relationships that were introduced.

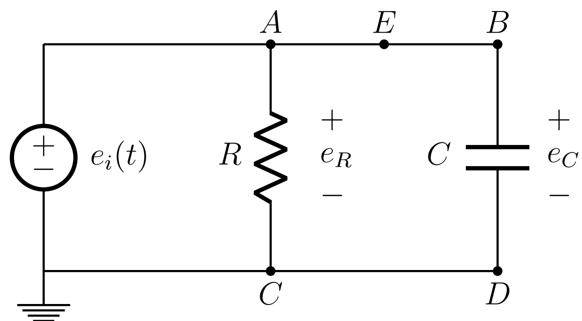


Figure 1: Networks Of Electrical Elements

An electrical circuit will consist of several such elements connected to form a **network**, with multiple pathways for current to flow through this network. Figure 1 shows such a network, consisting of three elements. These elements are connected together at A and C, although we could also define points E, B, and D for convenience.

4.1 Ideal Conductors

The analysis of the networks of elements like the one in Figure 1 follows a lumped-parameter assumption. We assume that the wires are ideal circuit elements, and possibly lump their resistance into a few individual resistors. With that assumption, if two points are connected by a wire, we can assume that

1. the resistance between them is zero, which means
2. the voltage at those points are equal

Another property of ideal conductors is that they don't store charge.

The assumption that ideal wires connect elements leads to Kirchoff's laws. Kirchoff's laws allow us to determine the answers to these questions, by using two principles:

- Kirchoff Current Law
- Kirchoff Voltage Law

4.2 Kirchoff's Current Law

Charge does not accumulate in the terminals/wires that connect circuit elements, because they are ideal conductors. If the charge at a point P is always zero, then it's rate of change is always zero. That is, the rate of charge (current) arriving at a point on an ideal conductor must equal the rate of charge (current) leaving that point.

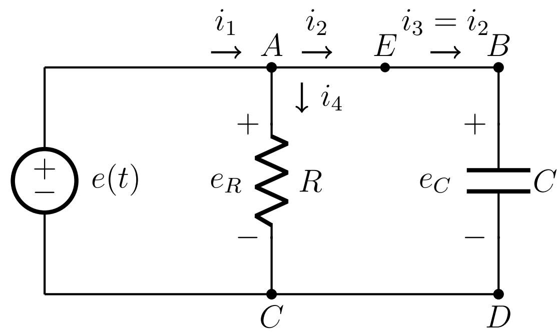


Figure 2: Kirchoff's Laws Demo Circuit

For example, at point E in Figure 2, the current i_2 entering E must equal the current i_3 leaving E. This principle seems obvious and uninteresting at such points. However, this principle has value at point A because it is a node, a point where multiple wires intersect.

Kirchoff's Current Law: The algebraic sum of the currents entering a node equals the algebraic sum of currents leaving that node.

At node A, applying the KCL gives the relationship $i_1 = i_2 + i_4$. Applying KCL at B and D gives us that The current in the wire connecting D and C is i_2.

What is the result of KCL applied to node C?

4.3 Kirchoff's Voltage Law

The voltage difference between points connected by an ideal conductor is zero. Therefore, the voltage difference between a point on an ideal conductor and itself is clearly zero. This principle implies that if we sum the voltages across a loop that begins at a point on an ideal conductor and ends on that point, that sum should equal zero.

Kirchoff's Voltage Law: The algebraic sum of voltages across elements in a closed circuit (a loop) is equal to zero.

Consider Figure 2. Clearly,

$$(e_A - e_C) + (e_C - a_A) = 0.$$

The point of the loop is to use different branches connecting A and C to replace the terms in the brackets, resulting in taking voltage differences across elements in a loop. For example, $e_A - e_C$ along the middle branch is just e_R. Along the left branch, we get $e_A - e_C = e_i$. So, the KVL across the left loop (corresponding to the left and middle branches between A and C) gives us $e_R - e_i = 0$.

In total, we can consider three loops:

- Loop 1 contains the voltage source and resistor
- Loop 2 contains the voltage source and capacitor
- Loop 3 contains the reistor and capacitor

Applying the KVL to these three loops, in the direction $A \to B \to D \to C \to A$ gives

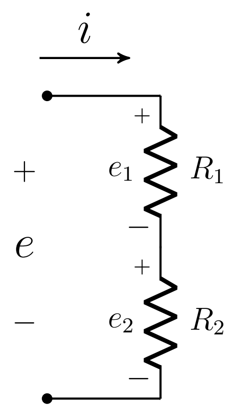
- Loop 1: $e_R - e_i = 0$
- Loop 2: $e_C - e_i = 0$
- Loop 3: $e_R - e_C = 0$

4.4 Voltage Divider Rule

For resistors R_1 and R_2 in series as shown, if the total voltage across them is e, then

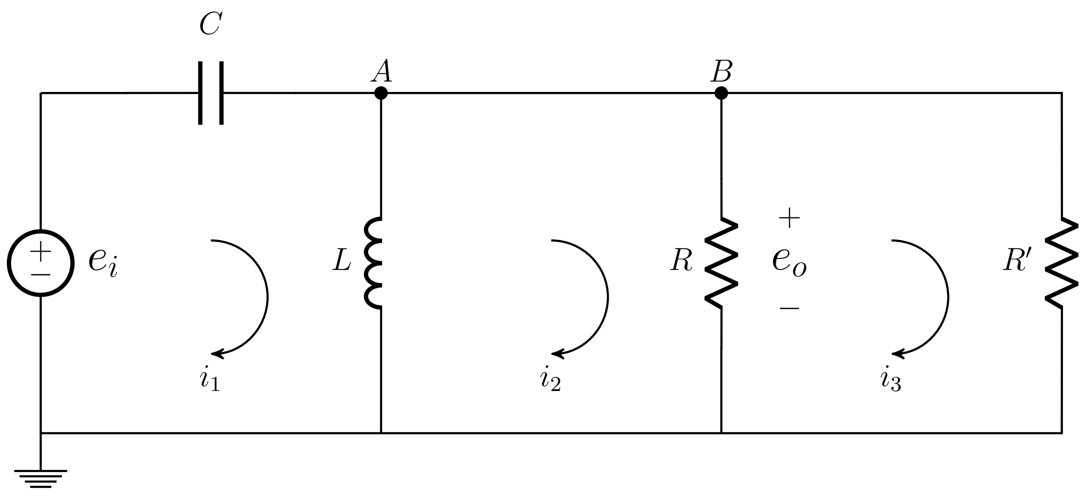
$$e_1 = \frac{R_1}{R_1 + R_2} e, \quad e_2 = \frac{R_2}{R_1 + R_2} e.$$

We may derive this rule by applying the KCL at a point in between the two resistors and noting that $e = e_1 + e_2 = i(R_1 + R_2)$.



4.5 Loop-Equation vs Node-Equation

Loop-Equation. Let i_1, i_2, and i_3 be the loop currents as shown.



Node-Equation. Let i'_1, i'_2, i'_3, i'_4, and i'_5 be the currents at nodes as shown.

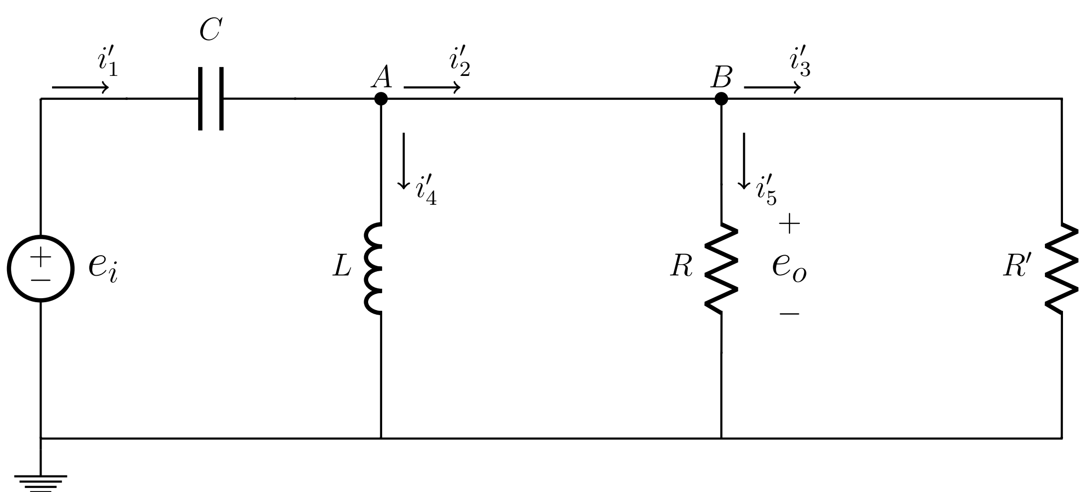


Table 1 above shows us that the loop equation method is ultimately equivalent to the node equation method, but just uses fewer variables from the beginning. As the number of branching points (nodes) increases, the number of variables in the node-equation method increases, making the loop-equation method simpler. When there's only one branching of the current from a source, the two methods are almost equivalent in terms of algebra required. The main advantage of the node-equation method is that you are forced to explicitly identify currents in each branch, unlike the loop-equation method, reducing the potential for mistakes. In other words, applying the loop-equation method consistently requires practice.

Element	Loop-Equation	Node-Equation
C	i_1	i'_1
wire AB	i_2	i'_2
R'	i_3	i'_3
L	$i_1 - i_2$	i'_4 $(= i'_1 - i'_2)$
R	$i_2 - i_3$	i'_5 $(= i'_2 - i'_3)$

Table 1: Relating the current variables in the loop-equation and node-equation methods.

4.6 General Solution Approach

To analyze the circuit, which is a network of elements connected by ideal conductors, we need to identify one of two things:

1. the independent nodes
2. the independent loops

Why ? The KVL applied to each loop will connect the voltages across different elements. Figure 1 has two three loops: the outer loop, left inner loop, and right inner loop. Only two of them are independent, since anything that we derive by applying the KVL to two of them can be combined to produce the same result of applying the KVL to the remaining one: there is no new information in that non-independent loop. In the section on KVL above, notice that adding the first two equations would produce the third.

The same issue arises when applying the KCL to nodes. Figure 1 has two important nodes: A and C. These are important because the current possibly branches at those points, but they are not independent. E, B, and D can be called nodes, but they aren't really important. For example, the voltage at E and B are the same, and the current through them is the same. Even though A and C are important, they are not independent. If we apply the KCL to Figure 2 to node A, we get

$$i_1 = i_2 + i_4.$$

The current from D to C is $i_3 = i_2$ the current from A to C is i_4. The current from C to the voltage source must be i_1, so that the KCL at C provides

$$i_1 = i_2 + i_4,$$

the same as that at A. So, this circuit has only one independent node, and two independent loops.

Summary: Apply the KCL at independent nodes, or the KVL to independent loops.

For Figure 2, we get the following relationships:

$$e_R - e_i = 0 \tag{1}$$
$$e_C - e_i = 0 \tag{2}$$
$$i_2 + i_4 = i_1 \tag{3}$$

Next. The KCL will relate currents. The KVL will relate voltages. If we need an input current related to an output current, perhaps all the equations from applying KCL (node-equation method) will allow us to derive that relationship. If we need an input voltage related to an output voltage, perhaps all the equations from applying KVL (loop-equation method) will allow us to derive that relationship. If our output was e_R, then we just get the relationship $e_R(t) = e_i(t)$. In practice, the node-equation method is usually a better method, and we'll see why in a minute.

What we don't seem to get from the network is a relationship between currents and voltages. What happens when we need relationships between an input voltage/current and an output current/voltage? What we do is to use the element relationships between voltages and currents to replace what we don't need with what we need. The same idea is what makes the node-equation method the best option in most cases, because the currents can be converted to voltages using element laws. Suppose we wanted to relate i_4 and e_i, we would use $e_R = i_4 R$ together with $e_R = e_i$, so that

$$i_4(t) = \frac{1}{R} e_i(t).$$

If we want to relate i_2 and input e_i, we do so in a few steps. The two voltage relationships imply that $e_C = e_R$, and for the capacitor $i_C = C\frac{d}{dt}e_C$, so that together

$$i_2(t) = C\frac{d}{dt}e_R(t).$$

How do we relate i_1 and e_C? The KCL says $i_1 = i_2 + i_4$. We know from the elemental laws that $i_2 = C\dot{e}_c$, and $i_4 = e_R/R$, but since $e_R = e_C$ by the KVL, we can write

$$i_1(t) = C\dot{e}_c(t) + \frac{1}{R} e_C(t).$$

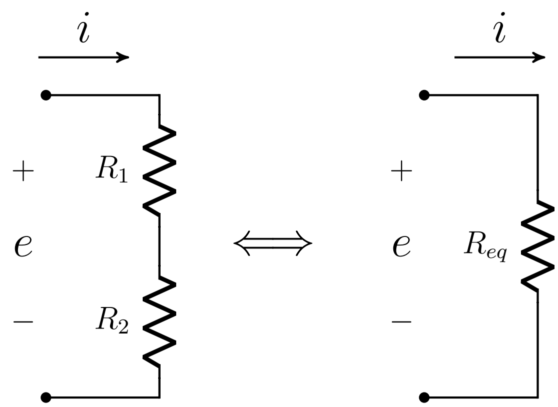
5 Circuit Elements in Series and Parallel

Lumped elements of the same type may be further lumped together when they are connected in series (only one terminal connected to that of other), or when they are connected in parallel (both terminals are connected).

5.1 Resistors in Series and Parallel

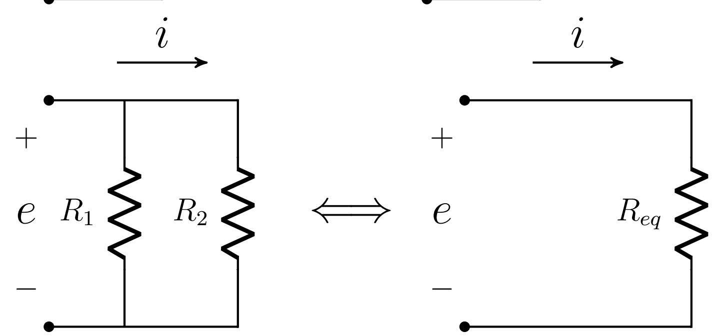
The resistors share only one terminal. They are in series.

$$R_{eq} = R_1 + R_2$$



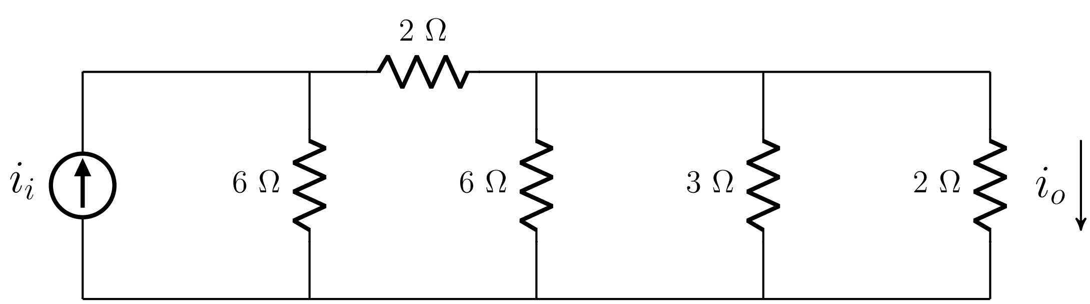
The resistors have one of each terminal connected to that of the other. They are in parallel.

$$R_{eq} = \frac{R_1 R_2}{R_1 + R_2}$$



Example 1. Find the equivalent resistance of the circuit below.

Find the value of i_o in terms of i_i. □



Solution : The equivalent resistance is 2Ω, and the relationship is $i_0 = \frac{1}{3} i_i$. (Work this problem out by yourself.)

5.2 Capacitors in Series and Parallel

The same analysis applied to capacitors, with the 'pattern' reversed.

Series: $C_{eq} = \frac{C_1 C_2}{C_1 + C_2}$

Parallel: $C_{eq} = C_1 + C_2$

6 Transfer Functions For Analysis

The previous sections derive ordinary differential equations using current-voltage relationships and Kirchoff's laws. Here, we take an alternative approach by starting from the complex s-domain from the beginning. The current-voltage laws define an electrical impedance. Impedance is a unified view of circuit elements, as described below.

Definition 1 (Electrical Impedance)**.** The electrical impedance $Z(s)$ of a two-terminal electrical element is the transfer function from current $i(t)$ flowing through it to the voltage $e(t)$ across its terminals:

$$Z(s) = \frac{\hat{e}(s)}{\hat{i}(s)} \tag{4}$$

Therefore, we would see that for resistor R, inductor L, and capacitor C:

- $Z_R(s) = R$
- $Z_L(s) = sL$
- $Z_C(s) = \frac{1}{sC}$

6.1 Impedances in Series and Parallel

The advantage of impedances is similar to the advantages of Laplace transforms: it turns calculus into algebra. Impedances are effectively generalized resistors.

- They obey the same series and parallel laws.
- They obey the voltage divider rule

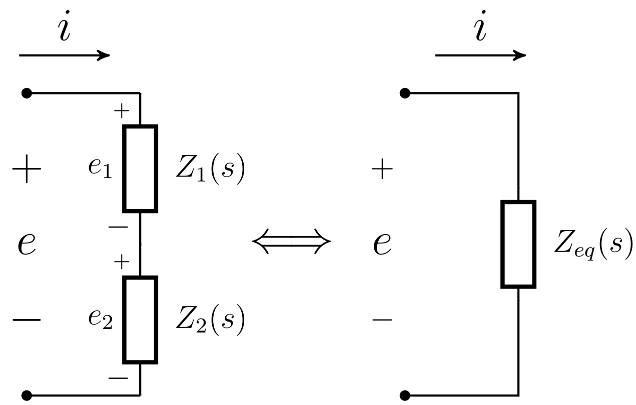
When the impedances $Z_1(s)$ and $Z_2(s)$ are in series, the equivalent impedance $Z_{eq}(s)$ is

$$Z_{eq}(s) = Z_1(s) + Z_2(s).$$

The voltage divider rule gives us:

$$\hat{e}_1(s) = \frac{Z_1(s)}{Z_1(s) + Z_2(s)}\hat{e}(s) \tag{5}$$

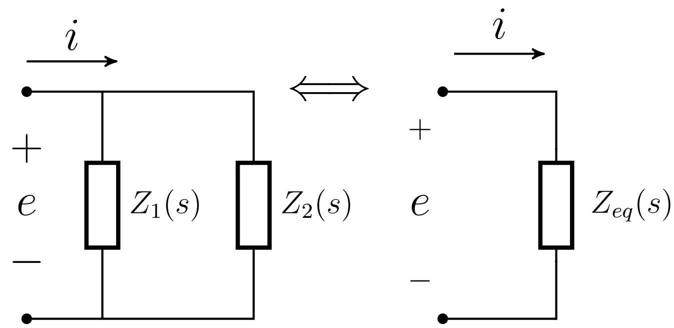
$$\hat{e}_2(s) = \frac{Z_2(s)}{Z_1(s) + Z_2(s)}\hat{e}(s) \tag{6}$$



When the impedances are in parallel, the equivalent impedance $Z_{eq}(s)$ is

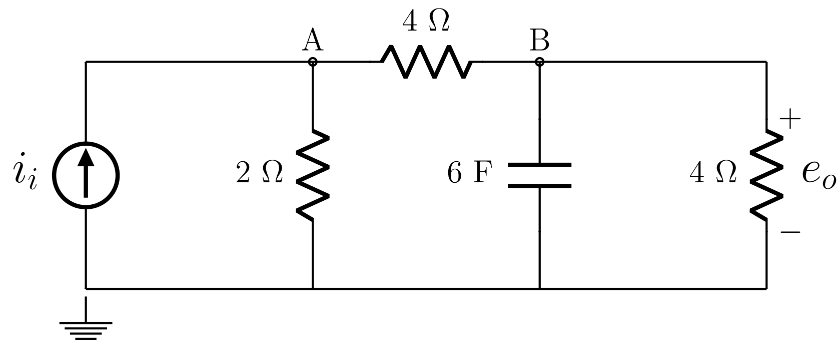
$$Z_{eq}(s) = \frac{Z_1(s)Z_2(s)}{Z_1(s) + Z_2(s)}.$$

The voltage across both impedances are equal to e.

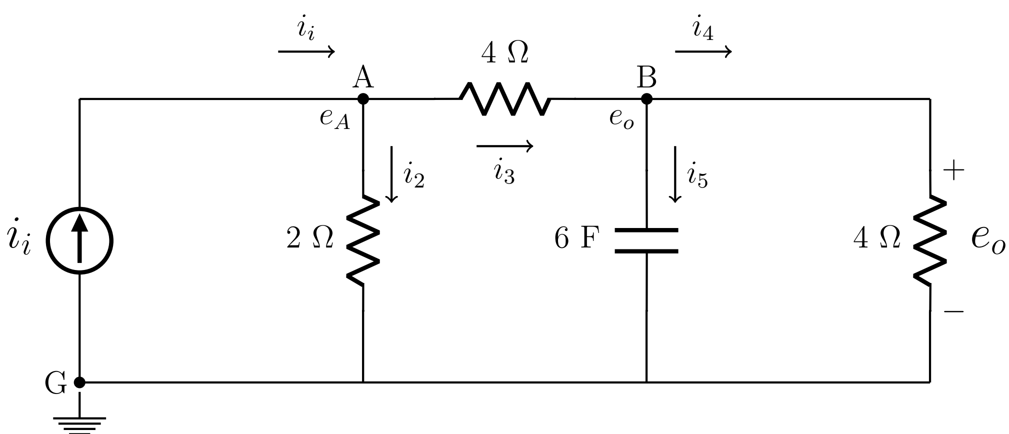


7 Examples

Example 2. Find an input-output model with input i_i and output e_o. □



Solution: Let's first annotate the diagram to help create a solution.



- Identify the **independent** nodes. These turn out to be A, B, and G.
- Identify voltages at each nodes, reusing provided information relevant to some nodes.
  - The voltage at G is taken to be zero, since it is directly connected to the ground.
  - Let e_A be the voltage at A.
  - The voltage at B is the same as $e_o(t)$, since the voltage at G is zero.
- Add symbols for currents flowing into and out of these nodes
  - The current flowing into A is i_i, so we don't need label it as i_1.

We're now ready to apply KCL laws. For each element, we figure out the voltage across the element in terms of the voltages we have defined, and apply the relationship between current and voltage for that type of element.

Apply the KCL at A:

$$i_i = i_2 + i_3 \tag{7}$$
$$\implies i_i = \underbrace{\frac{e_A}{2}}_{i=e/R} + \underbrace{\frac{e_A - e_o}{4}}_{i=e/R} \tag{8}$$
$$\implies 4i_i = 3e_A - e_o \tag{9}$$

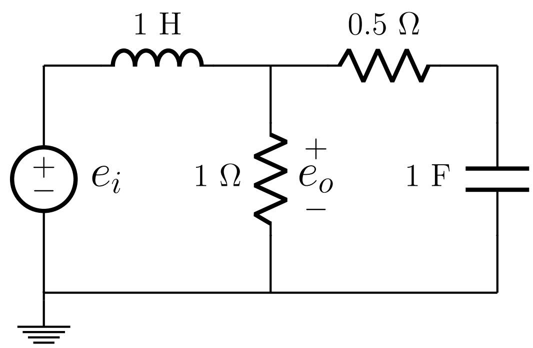
Apply KCL at B:

$$i_3 = i_4 + i_5 \tag{10}$$
$$\implies \underbrace{\frac{e_A - e_o}{4}}_{i=e/R} = \underbrace{\frac{e_o}{4}}_{i=e/R} + \underbrace{6\dot{e}_o}_{i=C\dot{e}} \tag{11}$$
$$\implies e_A = 24\dot{e}_o + 2e_o \tag{12}$$

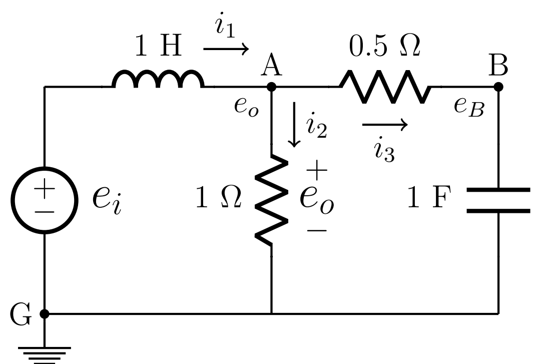
We substitute for the e_A in (9) using (12):

$$4i_i = 3e_A - e_o \tag{13}$$
$$\implies 4i_i = 3(24\dot{e}_o + 2e_o) - e_o \tag{14}$$
$$\implies 4i_i = 72\dot{e}_o + 5e_o \tag{15}$$

Example 3. Find an input-output model with input e_i and output e_o. □



Solution: We repeat the same steps.



- We have nodes A, B and G. The voltage at G is taken to be zero, so the voltage at A is e_o. Let the voltage at B be e_B.
- We identify three currents i_1, i_2, and i_3

Let's apply the KCL at A.

Apply the KCL at A:

$$i_1 = i_2 + i_3 \tag{16}$$

$$\implies \underbrace{i_1(0) + 1 \cdot \int_0^t (e_i - e_o)d\tau}_{e=L\frac{di}{dt} \implies Li=\int e d\tau} = \underbrace{\frac{e_o}{1}}_{i=e/R} + \underbrace{\frac{e_o - e_B}{0.5}}_{i=e/R} \tag{17}$$

$$\implies i_1(0) + \int_0^t (e_i - e_o)dt = \frac{3}{2}e_o - \frac{1}{2}e_B \tag{18}$$

$$\implies 2e_i - 2e_o = 3\dot{e}_o - \dot{e}_B \quad \text{(Differentiate to remove integral)} \tag{19}$$

Apply the KCL at B:

$$i_3 = i_3 \tag{20}$$

$$\implies \underbrace{\frac{e_o - e_B}{0.5}}_{i=e/R} = \underbrace{1 \cdot \dot{e}_B}_{i=C\dot{e}} \tag{21}$$

$$\implies e_o - e_B = 2\dot{e}_B \tag{22}$$

If we eliminate e_B from (19) and (22), we will obtain our desired input-output ODE. Rewriting the two-equations:

$$2e_i - 2e_o = 3\dot{e}_o - \dot{e}_B \tag{23}$$

$$e_o - e_B = 2\dot{e}_B \tag{24}$$

It's not clear how to eliminate e_B because of the derivative. This situation is where the advantage of the p-operator approach comes through.

$$2e_i - 2e_o = 3pe_o - pe_B \tag{25}$$

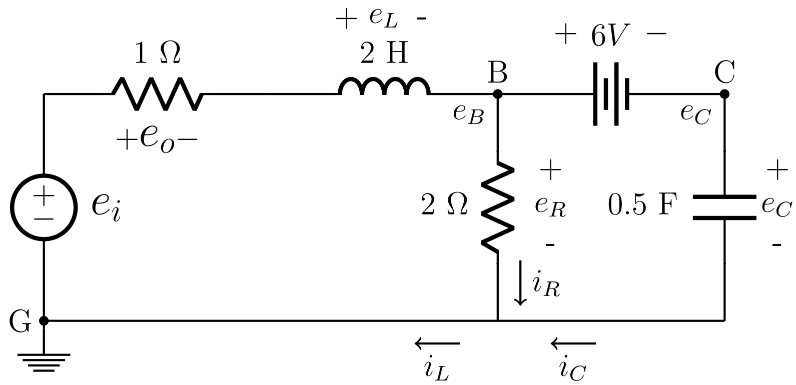
$$e_o - e_B = 2pe_B \implies e_B = \frac{1}{2p+1}e_o \tag{26}$$

$$\implies 2e_i - 2e_o = 3pe_o - p\left(\frac{1}{2p+1}e_o\right) \tag{27}$$

$$\implies 2(2p+1)e_i - 2(2p+1)e_o = (6p^2+3p)e_o - pe_o \tag{28}$$

$$\implies 2(2p+1)e_i = (6p^2+6p+2)e_o \tag{29}$$

$$\implies 3\ddot{e}_o(t) + 3\dot{e}_o(t) + e_o(t) = 2\dot{e}_i(t) + e_i(t) \tag{30}$$



Example 4 (State Variables)**.** Find a state-variable model with input e_i and output e_o. □

Solution:

Let the voltages across the inductor, capacitor, and 2Ω resistor be e_L, e_C, and $e_R = e_B$ respectively. Similarly, let the currents through them be i_L, i_C, and i_R.

The state of this system will be i_L and e_C. We know that $\frac{d}{dt}i_L = e_L/L$, and $\frac{d}{dt}e_C = i_C/C$. So, our goal will be to express i_C and e_L in terms of the state variables i_L and e_C. It makes sense to use KCL for i_L, and KVL for e_C.

The KCL at node B yields

$$\begin{aligned} i_L &= i_R + i_C \quad (31)\\ &= \frac{e_B}{2} + i_C \quad (32)\\ &= \frac{e_C+6}{2} + i_C \quad (33)\\ \implies i_C &= i_L - \frac{e_C}{2} - 3 \quad (34) \end{aligned}$$

Notice that we have state variables and inputs on the right hand side above.

KVL applied to the outer loop yields

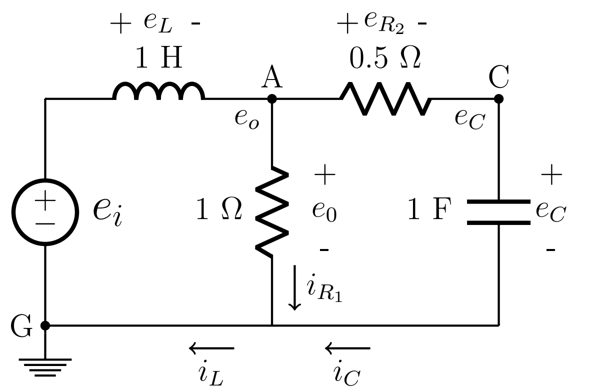
$$\begin{aligned} 0 &= e_i - e_o - e_L - 6 - e_C \quad (35)\\ \implies e_L &= e_i - e_o - 6 - e_C \quad (36)\\ \implies e_L &= e_i - i_L - 6 - e_C \quad (37) \end{aligned}$$

Again, we have state variables and inputs on the right hand side above.

Putting it all together :

$$\begin{aligned} \frac{di_L}{dt} = \frac{1}{L}e_L &\implies \frac{di_L}{dt} = -\frac{1}{2}i_L - \frac{1}{2}e_C + 3 + \frac{1}{2}e_i \quad (38)\\ \frac{de_C}{dt} = \frac{1}{C}i_C &\implies \frac{de_C}{dt} = 2i_L - e_C - 6 \quad (39)\\ e_o &= i_L \quad (40) \end{aligned}$$

Example 5. Find a state-variable model with input e_i and output e_o. □



Solution: The state of this system is taken as i_L, the current through the inductor and e_C, the voltage across the capacitor.

KCL at A:

$$i_L = i_C + i_{R_1} \tag{41}$$
$$= i_C + \frac{e_o}{1} \tag{42}$$
$$= \frac{e_o - e_C}{0.5} + e_o \tag{43}$$
$$\implies e_o = \frac{1}{3} i_L + \frac{2}{3} e_C \tag{44}$$
$$\text{Now, } i_C = \frac{e_o - e_C}{0.5} = \frac{2}{3} i_L - \frac{2}{3} e_C \quad \text{(using } e_o \text{ from above)} \tag{45}$$

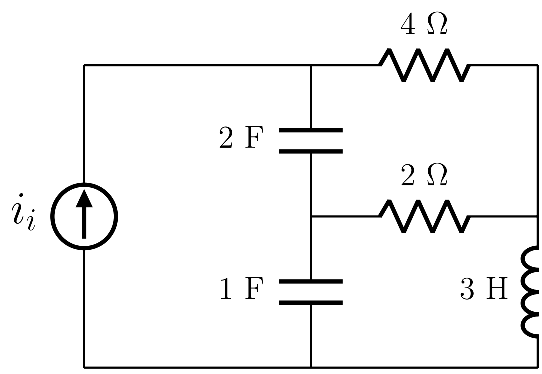
KVL on left loop:

$$0 = e_i - e_L - e_o \tag{46}$$
$$\implies e_L = e_i - e_o \tag{47}$$
$$\implies e_L = e_i - \frac{1}{3} i_L - \frac{2}{3} e_C \tag{48}$$

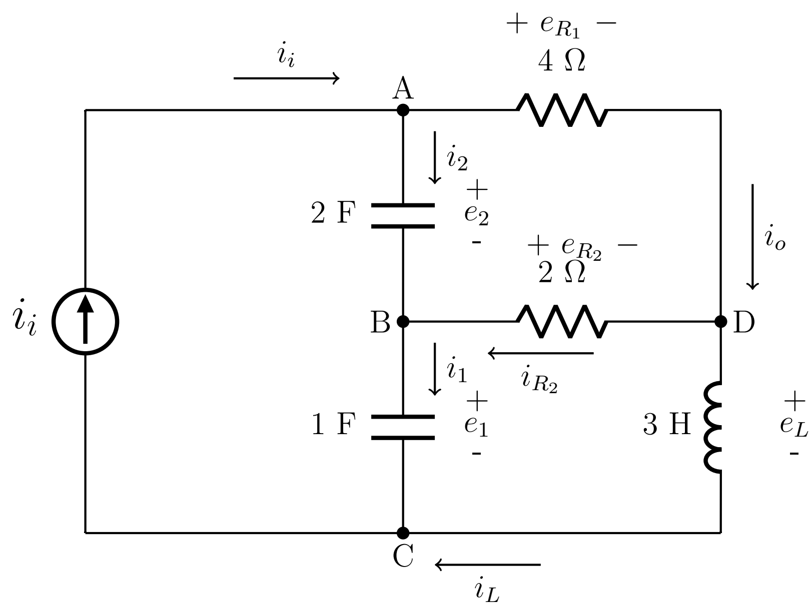
Our usual state-variable equations become

$$\frac{di_L}{dt} = \frac{1}{L} e_L \implies \frac{di_L}{dt} = -\frac{1}{3} i_L - \frac{2}{3} e_C + e_i \tag{49}$$
$$\frac{de_C}{dt} = \frac{1}{C} i_C \implies \frac{de_C}{dt} = \frac{2}{3} i_L - \frac{2}{3} e_C \tag{50}$$
$$e_o = \frac{1}{3} i_L + \frac{2}{3} e_C \tag{51}$$

Example 6. Find a state-variable model with input i_i and output i_o. □



Solution : We take the state to be i_L, e_1, and e_2.



We need to express e_L, i_1, and i_2 in terms of these states.

Apply KCL to to D:

$$i_0 = i_L + i_{R_2} \tag{52}$$
$$= i_L + \frac{e_L - e_1}{2} \tag{53}$$

Apply KCL to A, C:

$$i_i = i_L + i_1 \implies i_1 = i_i - i_L \tag{54}$$
$$i_i = i_0 + i_2 \implies i_2 = i_i - i_0 \tag{55}$$

Apply KVL to outer loop:

$$0 = e_1 + e_2 - e_3 - e_L \tag{56}$$
$$\implies 0 = e_1 + e_2 - 4i_0 - e_L \tag{57}$$
$$\implies 0 = e_1 + e_2 - 4\left(i_L + \frac{e_L - e_1}{2}\right) - e_L \tag{58}$$
$$\implies 0 = e_1 + e_2 - 4i_L - 2e_L + 2e_1 - e_L \tag{59}$$
$$\implies e_L = e_1 + \frac{1}{3}e_2 - \frac{4}{3}i_L \tag{60}$$

Solve for i_0:

$$i_0 = i_L + \frac{e_L - e_1}{2} \tag{61}$$
$$= i_L + \frac{e_L}{2} - \frac{e_1}{2} \tag{62}$$
$$= i_L + \frac{1}{2}\left(e_1 + \frac{1}{3}e_2 - \frac{4}{3}i_L\right) - \frac{e_1}{2} \tag{63}$$
$$= \frac{1}{3}i_L + \frac{1}{6}e_2 \tag{64}$$

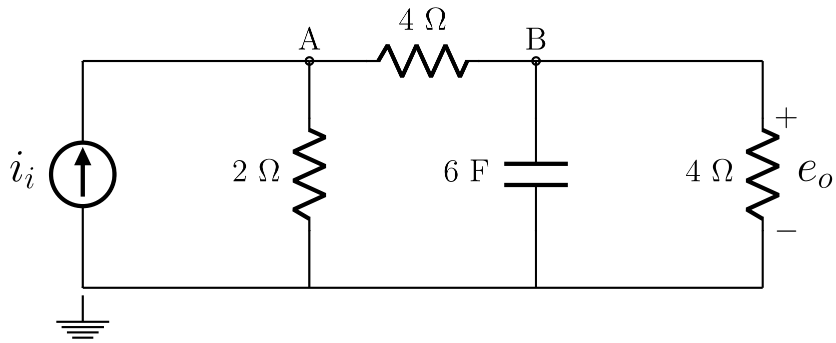
So that

$$i_2 = i_i - i_0 = -\frac{1}{3}i_L - \frac{1}{6}e_2 + i_i.$$

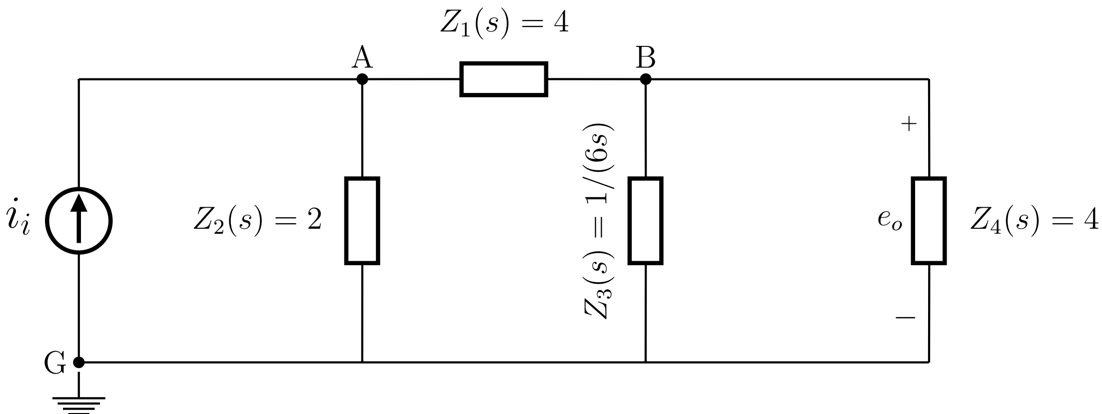
Putting it all together, we get

$$\frac{di_L}{dt} = \frac{1}{3}e_1 + \frac{1}{9}e_2 - \frac{4}{9}i_i \tag{65}$$
$$\frac{de_1}{dt} = -i_L + i_i \tag{66}$$
$$\frac{de_2}{dt} = -\frac{1}{12}e_2 - \frac{1}{6}i_L + \frac{1}{2}i_i \tag{67}$$
$$i_0 = \frac{1}{6}e_2 + \frac{1}{3}i_L \tag{68}$$

Example 7 (Example 2). Find an input-output model with input i_i and output e_o. □

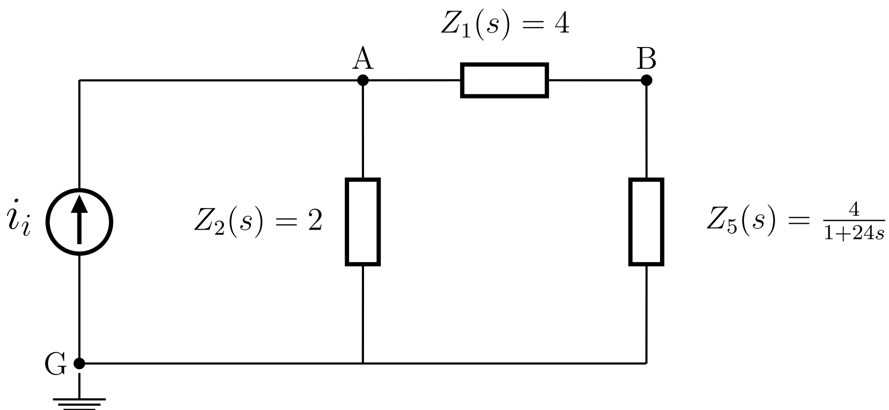


Solution: Let's first annotate the diagram to help create a solution.



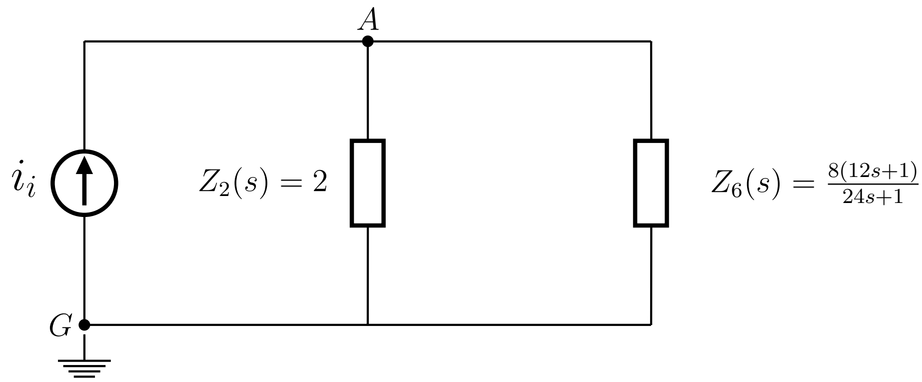
We replace $Z_3(s)$ and $Z_4(s)$ with a single impedance given by

$$Z_5(s) = \frac{Z_3(s)Z_4(s)}{Z_3(s) + Z_4(s)} = \frac{\frac{1}{6s} \cdot 4}{\frac{1}{6s} + 4} = \frac{4}{1 + 24s} \tag{69}$$



We replace $Z_1(s)$ and $Z_5(s)$ with a single impedance given by

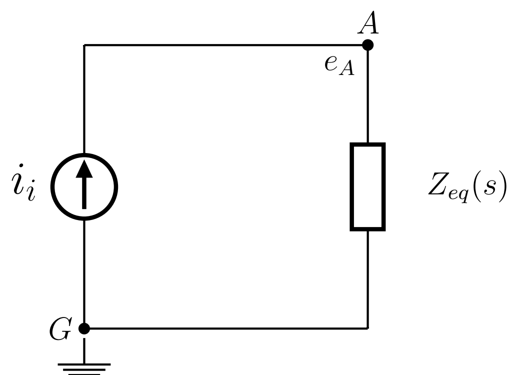
$$Z_6(s) = 4 + \frac{4}{1 + 24s} = \frac{8(12s + 1)}{24s + 1} \tag{70}$$



Finally, we have

We replace $Z_2(s)$ and $Z_6(s)$ with a single impedance given by

$$Z_{eq}(s) = \frac{2 \cdot \frac{8(12s+1)}{24s+1}}{2 + \frac{8(12s+1)}{24s+1}} = \frac{8(12s+1)}{24s+1+4(24s+1)} = \frac{8(12s+1)}{72s+5} \tag{71}$$



It's now easy to read off the following relationship:

$$e_A(s) = Z_{eq}(s)\hat{i}_i(s) \tag{72}$$

By the voltage divider rule (see Section 4.4),

$$
\begin{aligned}
e_B(s) &= \frac{Z_1(s)}{Z_1(s)+Z_5(s)} e_A(s) && && (73)\\
\Longrightarrow \quad e_o(s) &= \frac{Z_1(s)}{Z_1(s)+Z_5(s)} e_A(s) && (e_o = e_B) && (74)\\
\Longrightarrow \quad e_o(s) &= \frac{Z_1(s)}{Z_1(s)+Z_5(s)} Z_{eq}(s)\hat{i}_i(s) && \text{from } (72) && (75)\\
&= \frac{4}{4+\frac{4}{24s+1}} Z_{eq}(s)\hat{i}_i(s) && && (76)\\
&= \frac{1}{2(12s+1)} Z_{eq}(s)\hat{i}_i(s) && && (77)\\
&= \frac{1}{2(12s+1)} \frac{8(12s+1)}{72s+5}\hat{i}_i(s) && && (78)\\
&= \frac{4}{72s+5}\hat{i}_i(s) && && (79)
\end{aligned}
$$

$$
\Longrightarrow \quad 4i_i(s) = (72s+5)e_o(s) \tag{80}
$$

From the Laplace inverse,

$$
4i_i(t) = 72\dot{e}_o(t) + 5e_o(t) \tag{81}
$$

which you may compare with (15).

Notice that we did not apply Kirchoff's laws explicitly, nor did we apply the p-operator to obtain the input-output equation.